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RADIATIVE TRANSPORT AND WALL TEMPERATURE SLIP IN AN ABSORBING PLANAR MEDIUM

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Abstract—Radiative heat transfer through a nonisothermal absorbing and emitting grey gas between heated walls is studied. Specific attention is directed toward the evaluation of temperature near the walls and of the precise evaluation of energy flux by means of methods and tabulated functions studied by Chandrasekhar and Ambartsumian. The precision achieved permits an assessment of the accuracy of existing approximate methods and of the errors incurred in numerical solutions of the governing equations.

NOMENCLATURE

$E_n(x)$, integroexponential function of order n (see equation (2));
$H(\mu)$, auxiliary function introduced in equation (50);
k, volumetric absorption coefficient;
L, geometric thickness of plane layer;
$L(\xi, \xi_1; \xi_L)$, resolvent kernel (see equation (20));
q, net rate of energy transport per unit area, $q^+ - q^-$;
q^+, q^-, half-range fluxes defined following equation (3);
Q, Q_s, dimensionless fluxes introduced in equation (18);
S, internal heat source per unit volume per unit time;
T, temperature, absolute;
x, geometric depth in absorbing layer;
$X(\mu, \xi_L)$, Chandrasekhar's X function (see equation (30a));
$Y(\mu, \xi_L)$, Chandrasekhar's Y function (see equation (30b));
$\alpha_n(\xi_L)$, nth moment of X function (see equation (31a));
$\beta_n(\xi_L)$, nth moment of Y function (see equation (31b));
ϵ, surface emissivity;
$\Theta(\xi)$, $\Theta_s(\xi)$, universal functions introduced in equation (9) (see also equation (12));
ξ, optical depth in absorbing layer $d\xi = k\,dx$;
σ, Stefan–Boltzmann constant;
$\Phi(\xi)$, resolvent kernel evaluated at $\xi_1 = 0$, i.e. $L(0, \xi; \xi_L)$;
$\Psi(\xi, \xi_L)$, function defined by equation (42).

Subscripts
$w1$, evaluated at left wall;
$w2$, evaluated at right wall.

INTRODUCTION

THIS PAPER adds to the already considerable amount of literature on the transport of thermal radiation through an absorbing and emitting medium contained between heated, opaque walls. Its principal aim is to provide a standard of accuracy for those predictions that at the present time are most susceptible to numerical errors. In particular, attention is directed toward the evaluation of wall temperature slip, a discontinuity effect that occurs for finite values of optical thickness when energy transport is characterized solely as a radiant phenomenon. This objective follows immediately through use of methods developed by Ambartsumian and Chandrasekhar for use in radiation analysis. The present paper relates wall conditions to quadratures of functions which have previously been calculated to a considerable degree of accuracy. Additional labor is thus precluded through the use of existing tables and, at the same time, a check is provided whereby approximate methods may be judged.

In the problem considered here two parallel, opaque walls, held at different temperatures, are assumed to radiate isotropically. The wall emissivities are not necessarily the same but they are taken to be independent of radiation wave length. The intervening medium has similar properties: it is isotropic, it radiates diffusely, and its emissivity is expressed by an appropriate average over the frequency spectrum. A number of publications have dealt specifically with this physical case; the review article by Viskanta and Grosh [1] furnishes a ready survey of available calculations. It is difficult to give an adequate chronological development of the theory and results, partly because of analogies and interrelations existing between similar considerations arising in the study of stellar and planetary atmospheres as well as in the study of neutron transport and molten translucent materials. The basic theory of radiative transport is developed in the books of Chandrasekhar [2], Kourganoff [3], and Sobolev [4]. Specific adaptations to the interests of the mechanical and thermal engineer are to be found in the papers of Viskanta and Grosh [5], Usiskin and Sparrow [6], Meghreblian [7], and the two authors Howell and Perlmutter [8, 9]. Usiskin and Sparrow appear to be the first to give detailed solutions for a uniformly generating medium between black walls. They pointed out the applicability of their results to the case of nonblack walls. Perlmutter and Howell gave a precise synthesis of the use of the black wall solutions and used a Monte Carlo method to carry out their numerical calculations.

The computational problem can be resolved into one of solving uncoupled integral equations [see equations (11a) and (11b)]. A wide diversity of methods exists and the current literature displays this freedom of choice. The thorough study of different techniques is fully exemplified in the treatise by Kourganoff [3] but his extensive investigation is held within reasonable bounds only after attention is limited to a semi-infinite medium. Sobolev [4] has systematically pursued the objective of calculating the resolvent kernel of an integral equation with a known kernel and the present paper uses this approach. The actual calculation of the resolvent kernel remains a computational task but the methods introduced originally by Ambartsumian and developed in [2], [3], and [4] provide a direct way to evaluate conditions on the boundary of the medium. This is especially fortuitous for the problem at hand since virtually all other available techniques suffer their maximum inaccuracy at the boundary. The smoothing property of integrals allows remarkably good approximations to be achieved with small effort. It is known, for example, that linear approximations to the emission function predict flux with an error of, at most, 2 or 3 per cent. This suggests that standard iterative techniques should converge rather rapidly. A cursory study of the integral equations shows, however, that in the immediate vicinity of the walls the emission function is a nonanalytical function of distance from the wall. The accuracy of the convergence in this vicinity is thus not easy to estimate and the availability of a check on end conditions is of mathematical as well as physical interest.

GENERAL ANALYSIS

Governing equations

As indicated in Fig. 1 we consider two walls of infinite lateral extent and a distance L apart. The single coordinate x is measured relative to the left wall. Known physical conditions include the temperatures and emissivity coefficients of the two walls and the absorptivity of the intervening medium. Thus, using the subscripts $w1$ and $w2$ to denote conditions at the walls at $x = 0$ and $x = L$, respectively, we have given T_{w1}, T_{w2}, ϵ_{w1}, ϵ_{w2} as well as k, the volumetric absorption coefficient of the medium. In addition, the possibility of heat generation within the medium is included and to this end we assume known the source function S which determines the energy generated per unit volume per unit time. We shall limit ourselves always to the special case for which S/k is uniform, the so-called constant source case. It

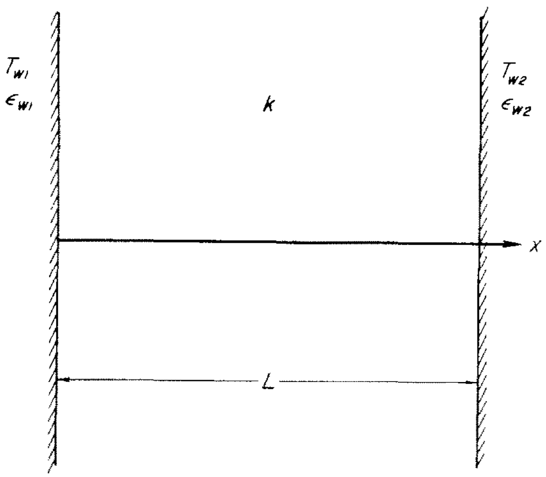

FIG. 1. Parallel walls separated by absorbing medium.

remains to predict the temperature distribution through the medium as well as the heat flux, that is, radiative transport of thermal energy per unit area per unit time.

The integral equation that fixes the temperature distribution is (see, e.g. [10])

$$\sigma T^4(\xi) = \frac{q_{w1}^+}{2} E_2(\xi) + \frac{q_{w2}^-}{2} E_2(\xi_L - \xi) + \frac{S}{4k} + \frac{1}{2} \int_0^{\xi_L} \sigma T^4(\xi_1) E_1(|\xi - \xi_1|) \, d\xi_1 \quad (1)$$

In this equation and subsequently, optical path length ξ is used instead of physical distance x. The relation $\xi = \int_0^x k \, dx$ holds and when $x = L$, $\xi = \xi_L$ where ξ_L is the so-called optical thickness of the plane layer of absorbing material. Also, σ is Stefan–Boltzmann's constant and the functions $E_1(\xi)$, $E_2(\xi)$ are defined by

$$E_n(\xi) = \int_0^1 \exp[-\xi/\mu] \, \mu^{n-2} \, d\mu = \int_1^\infty \frac{\exp[-\xi x_1] \, dx_1}{x_1^n} \quad (2)$$

The remaining undefined terms are associated with the concept of "half-range" fluxes. Flux $q(\xi)$ is, by definition

$$q(\xi) = q^+(\xi) - q^-(\xi) \quad (3)$$

where $q^+(\xi)$ is the energy per unit time emerging from the right face of a unit area at ξ and $q^-(\xi)$ is energy per unit time emerging from the left face. The half-range fluxes $q^+(\xi)$ and $q^-(\xi)$ are reckoned positively in the positive and negative ξ directions, respectively, so that $q(\xi) > 0$ corresponds to a net energy flow in the positive ξ direction.

Additional theoretical complexity results from the fact that equation (1) does not evolve in a form written explicitly in terms of the given physical parameters, that is, the wall temperatures and emissivities. Boundary conditions are required and for opaque walls these take the form

$$\left. \begin{aligned} q_{w1}^+ &= \epsilon_{w1} \sigma T_{w1}^4 + (1 - \epsilon_{w1}) q_{w1}^- \\ q_{w2}^- &= \epsilon_{w2} \sigma T_{w2}^4 + (1 - \epsilon_{w2}) q_{w2}^+ \end{aligned} \right\} \quad (4)$$

Equations (4) are energy balances that equate total energy leaving the walls to wall emission plus the reflected portion of the incoming energy flux. It is not necessary to assume opaqueness but only under this condition can the terms in parentheses in equations (4) be related directly to emissivity.

Setting $q_{w1} = q_{w1}^+ - q_{w1}^-$, we find, after algebraic manipulation of equations (4),

$$\left. \begin{aligned} q_{w1} \left(\frac{1}{\epsilon_{w1}} - 1 \right) &= \sigma T_{w1}^4 - q_{w1}^+ \\ q_{w2} \left(\frac{1}{\epsilon_{w2}} - 1 \right) &= -\sigma T_{w2}^4 + q_{w2}^- \end{aligned} \right\} \quad (5)$$

A companion relation to equation (1) is provided by the formula for flux; namely,

$$q(\xi) = 2q_{w1}^+ E_3(\xi) - 2q_{w2}^- E_3(\xi_L - \xi) + 2 \int_0^{\xi_L} \sigma T^4(\xi_1) \operatorname{sgn}(\xi - \xi_1) E_2(|\xi - \xi_1|) \, d\xi_1 \quad (6)$$

From equations (6) and (1) one deduces

$$\frac{dq(\xi)}{d\xi} = \frac{S}{k} \quad (7)$$

It follows that radiative flux is constant between the two walls, independent of position, when

$S = 0$. In general, integration of equation (7) yields

$$q_{w2} - q_{w1} = \int_0^{\xi_L} (S/k)\,\mathrm{d}\xi \tag{8}$$

Reduction to canonical form

The transformation

$$\sigma T^4(\xi) - q_{w2}^- = (q_{w1}^+ - q_{w2}^-)\,\Theta(\xi) + \frac{S}{k}\Theta_s(\xi) \tag{9}$$

is now introduced and equation (1) then becomes

$$\left.\begin{aligned}(q_{w1}^+ - q_{w2}^-)\,\Theta(\xi) + \frac{S}{k}\Theta_s(\xi) \\ = \left(\frac{q_{w1}^+ - q_{w2}^-}{2}\right) E_2(\xi) + \frac{S}{4k} \\ + \left(\frac{q_{w1}^+ - q_{w2}^-}{2}\right)\int_0^{\xi_L} \Theta(\xi_1)\,E_1(|\xi - \xi_1|)\,\mathrm{d}\xi_1 \\ + \frac{S}{2k}\int_0^{\xi_L} \Theta_s(\xi_1)\,E_1(|\xi - \xi_1|)\,\mathrm{d}\xi_1\end{aligned}\right\} \tag{10}$$

The linearity of equation (10) permits one to express its solution in terms of solutions of two independent integral equations:

$$\Theta(\xi) = \tfrac{1}{2}E_2(\xi) + \tfrac{1}{2}\int_0^{\xi_L} \Theta(\xi_1)\,E_1(|\xi - \xi_1|)\,\mathrm{d}\xi_1 \tag{11a}$$

$$\Theta_s(\xi) = \tfrac{1}{4} + \tfrac{1}{2}\int_0^{\xi_L} \Theta_s(\xi_1)\,E_1(|\xi - \xi_1|)\,\mathrm{d}\xi_1 \tag{11b}$$

The functions $\Theta(\xi)$ and $\Theta_s(\xi)$ are universal functions for the present planar problems and from them the solution for arbitrary boundary conditions can be calculated. Equations (11) are independent of the parameters affecting the particular conditions to be specified and this permits a physical interpretation of the universal functions in terms of black wall conditions, that is, wall emissivities such that $\epsilon_{w1} = \epsilon_{w2} = 1$. Starting initially with black wall conditions and with $S = 0$, the function

$$\left[\frac{\sigma T^4(\xi) - \sigma T_{w2}^4}{\sigma T_{w1}^4 - \sigma T_{w2}^4}\right]_{\substack{\epsilon_{w1}=\epsilon_{w2}=1 \\ S=0}} = \Theta(\xi) \tag{12a}$$

satisfies equation (11a). Similarly, if $S \neq 0$, $T_{w1} = T_{w2}$, and $\epsilon_{w1} = \epsilon_{w2} = 1$, the function

$$\left[\frac{\sigma T^4(\xi) - \sigma T_{w2}^4}{S/k}\right]_{\substack{\epsilon_{w1}=\epsilon_{w2}=1 \\ T_{w1}=T_{w2}}} = \Theta_s(\xi) \tag{12b}$$

satisfies equation (11b). Attention can therefore be limited to the case of black walls insofar as the integral equations (11a) and (11b) are concerned. It remains, however, to give explicit expressions for temperature distribution and flux in terms of $\Theta(\xi)$, $\Theta_s(\xi)$, and the basic physical parameters.

From equations (5) and (8)

$$q_{w1}^+ - q_{w2}^- = (\sigma T_{w1}^4 - \sigma T_{w2}^4) - \left(\frac{1}{\epsilon_{w1}} + \frac{1}{\epsilon_{w2}} - 2\right) q_{w1} - \left(\frac{1}{\epsilon_{w2}} - 1\right)\frac{S}{k}\xi_L \tag{13}$$

Equations (6) and (9) yield

$$q_{w1} = (q_{w1}^+ - q_{w2}^-)\left[1 - 2\int_0^{\xi_L} \Theta(\xi_1)\,E_2(\xi_1)\,\mathrm{d}\xi_1\right] - 2\frac{S}{k}\int_0^{\xi_L} \Theta_s(\xi_1)\,E_2(\xi_1)\,\mathrm{d}\xi_1 \tag{14}$$

These latter relations combine to give

$$q_{w1}^+ - q_{w2}^- = \frac{(\sigma T_{w1}^4 - \sigma T_{w2}^4) + \dfrac{S}{k}\left[-\left(\dfrac{1}{\epsilon_{w2}} - 1\right)\xi_L + 2\left(\dfrac{1}{\epsilon_{w1}} + \dfrac{1}{\epsilon_{w2}} - 2\right)\displaystyle\int_0^{\xi_L} \Theta_s(\xi_1)\,E_2(\xi_1)\,\mathrm{d}\xi_1\right]}{1 + \left(\dfrac{1}{\epsilon_{w1}} + \dfrac{1}{\epsilon_{w2}} - 2\right)\left[1 - 2\displaystyle\int_0^{\xi_L} \Theta(\xi_1)\,E_2(\xi_1)\,\mathrm{d}\xi_1\right]} \tag{15}$$

Since from equations (5), (8) and (9)

$$\sigma T^4 - \sigma T_{w2}^4 = (q_{w1}^+ - q_{w2}^-)\, \Theta(\xi) + \left(\frac{1}{\epsilon_{w2}} - 1\right) q_{w1} + \frac{S}{k}\left[\Theta_s(\xi) + \left(\frac{1}{\epsilon_{w2}} - 1\right)\xi_L\right] \tag{16}$$

we may combine the last three expressions into the desired relation in which temperature distribution is given in terms of T_{w1}, T_{w2}, ϵ_{w1}, ϵ_{w2}, ξ, ξ_L, S/k, $\Theta(\xi)$, and $\Theta_s(\xi)$. Finally one has,

$$\sigma T^4 - \sigma T_{w2}^4 = \frac{(\sigma T_{w1}^4 - \sigma T_{w2}^4)\left[\Theta(\xi) + \left(\frac{1}{\epsilon_{w2}} - 1\right) Q\right]}{1 + \left(\frac{1}{\epsilon_{w1}} + \frac{1}{\epsilon_{w2}} - 2\right) Q} + \frac{S}{k}\left\{\Theta_s(\xi) + \frac{\left[Q_s\left(\frac{1}{\epsilon_{w1}} + \frac{1}{\epsilon_{w2}} - 2\right) - \left(\frac{1}{\epsilon_{w2}} - 1\right)\xi_L\right]\Theta(\xi) + \left(\frac{1}{\epsilon_{w2}} - 1\right)\left(\xi_L - Q_s\right) + \xi_L\left(\frac{1}{\epsilon_{w1}} - 1\right)\left(\frac{1}{\epsilon_{w2}} - 1\right) Q}{1 + \left(\frac{1}{\epsilon_{w1}} + \frac{1}{\epsilon_{w2}} - 2\right) Q}\right\} \tag{17}$$

$$q_{w1} = \frac{(\sigma T_{w1}^4 - \sigma T_{w2}^4)\, Q - \frac{S}{k}\left[Q_s + \xi_L\left(\frac{1}{\epsilon_{w2}} - 1\right) Q\right]}{1 + \left(\frac{1}{\epsilon_{w1}} + \frac{1}{\epsilon_{w2}} - 2\right) Q} \tag{18}$$

where we have introduced the notation

$$Q = 1 - 2\int_0^{\xi_L} \Theta(\xi_1)\, E_2(\xi_1)\, d\xi_1$$

$$Q_s = 2\int_0^{\xi_L} \Theta_s(\xi_1)\, E_2(\xi_1)\, d\xi_1$$

SOLUTIONS

Figures 2 and 3 give solutions to equations (11) for a range of optical thicknesses. As remarked previously, comparable results have already been published. The present curves were calculated to a degree of accuracy in excess of the line width used. The values of the functions at the walls were checked numerically to three significant figures and involved discernible differences from some earlier results for which less precision was attempted. The curves may prove useful as a standard in the evaluation of methods for which it is difficult to assess rapidity of convergence. Indirect calculations involving random walk or Monte Carlo concepts, for example, or of arbitrarily imposed variational conditions, may be especially useful in extension to other geometries or conditions, yet need some estimate of residual errors.

The problem at hand dictates consideration of the conventional Fredholm integral equation

$$F(\xi) = G(\xi) + \tfrac{1}{2}\int_0^{\xi_L} F(\xi_1)\, E_1(|\xi - \xi_1|)\, d\xi_1 \tag{19}$$

along with its assumed inversion

$$F(\xi) = G(\xi) + \int_0^{\xi_L} G(\xi_1)\, L(\xi, \xi_1; \xi_L)\, d\xi_1 \tag{20}$$

Equation (19) has been written with a kernel to conform with both equations (11a) and (11b). The function $G(\xi)$ will, obviously, be equated later to $E_2(\xi)/2$ and 1/4 to get the solutions of interest. In equation (20), $L(\xi, \xi_1; \xi_L)$ is the resolvent kernel with dependence on the parameter ξ_L and is independent of $G(\xi)$.

We shall recount here sufficient information about the resolvent kernel to predict the desired physical quantities. A formal presentation of

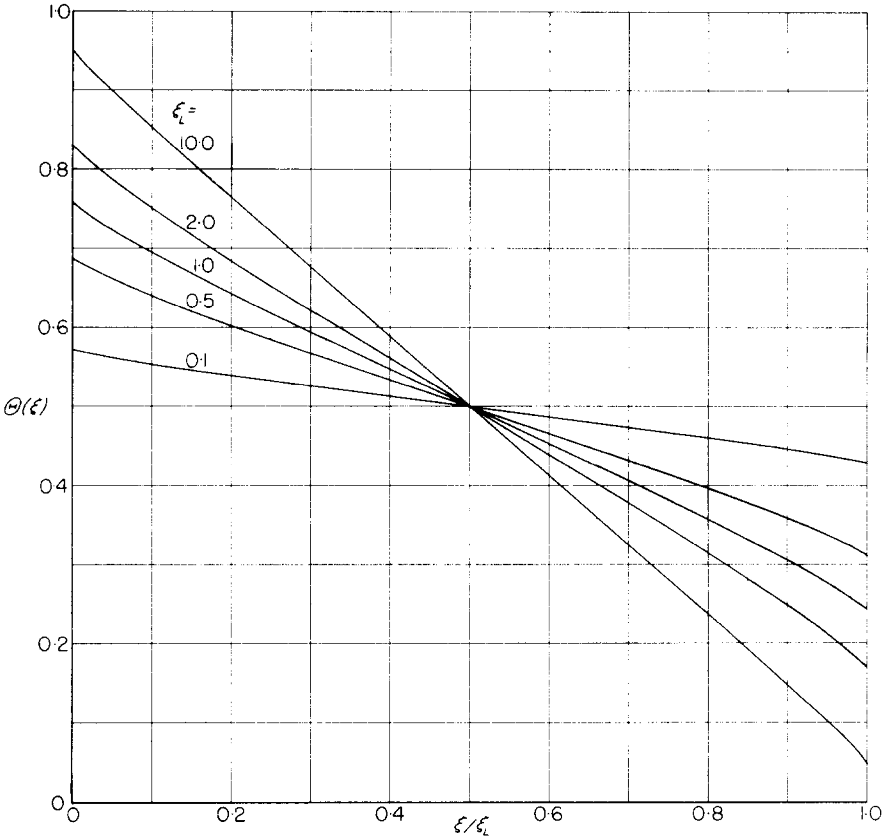

FIG. 2. Universal function $\Theta(\xi)$ for different values of optical thickness.

the theory can be found, for example, in [4, p. 299 *et seq.*]. Direct calculation yields

$$\frac{\partial L}{\partial \xi} + \frac{\partial L}{\partial \xi_1} = \Phi(\xi_1)\,\Phi(\xi) - \Phi(\xi_L - \xi_1)\,\Phi(\xi_L - \xi) \quad (21)$$

where

$$\Phi(\xi) = L(0, \xi; \xi_L) \quad (22)$$

From equation (21), one has, for $\xi_1 > \xi$

$$L(\xi_1, \xi; \xi_L) = \Phi(\xi_1 - \xi) + \int_0^{\xi} [\Phi(x)\,\Phi(x + \xi_1 - \xi) - \Phi(\xi_L - x)\,\Phi(\xi_L - x - \xi_1 + \xi)]\,\mathrm{d}x \quad (23)$$

which, together with the symmetry property,

$$L(\xi, \xi_1; \xi_L) = L(\xi_1, \xi; \xi_L) \quad (24)$$

expresses the resolvent in terms of the unidimensional function $\Phi(\xi)$ with the parameter ξ_L.

The function $\Phi(\xi)$ can be calculated as a solution of the equation

$$\Phi(\xi) = \tfrac{1}{2} E_1(\xi) + \tfrac{1}{2}\int_0^{\xi_L} \Phi(\xi_1)\,E_1(|\xi - \xi_1|)\,\mathrm{d}\xi_1 \quad (25)$$

We choose, however, to express it in terms of auxiliary functions. To this end, the function

$$F(\xi, \mu) = \mathrm{e}^{-\xi/\mu} + \tfrac{1}{2}\int_0^{\xi_L} F(\xi_1, \mu)\,E_1(|\xi - \xi_1|)\,\mathrm{d}\xi_1 \quad (26)$$

is introduced together with its inversion

$$F(\xi, \mu) = \mathrm{e}^{-\xi/\mu} + \int_0^{\xi_L} \exp\,[-\,\xi_1/\mu]\,L(\xi, \xi_1; \xi_L)\,\mathrm{d}\xi_1 \quad (27)$$

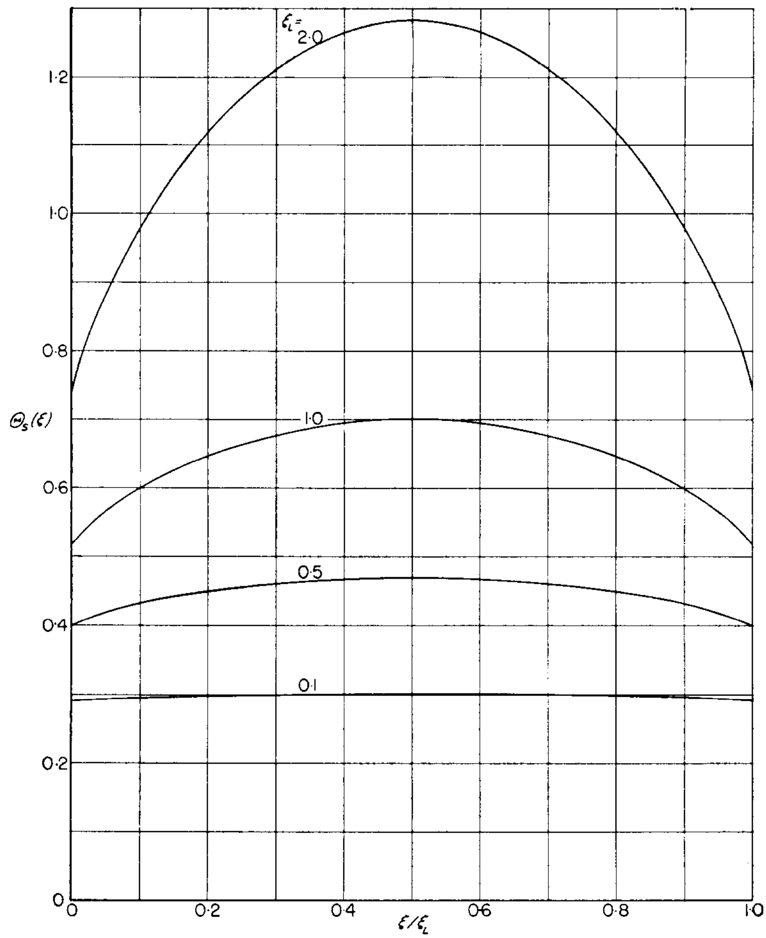

FIG. 3. Universal function $\Theta_s(\xi)$ for different values of optical thickness.

From equation (26) one deduces

$$\int_0^1 F(\xi, \mu)\, d\mu + \int_0^1 F(\xi_L - \xi, \mu)\, d\mu = 2 \quad (28)$$

Setting $\xi = 0$ and using equation (27) to evaluate this last relation, we get

$$\int_0^1 X(\mu, \xi_L)\, d\mu + \int_0^1 Y(\mu, \xi_L)\, d\mu = 2 \quad (29)$$

where

$$X(\mu, \xi_L) = 1 + \int_0^{\xi_L} \Phi(\xi_1) \exp[-\xi_1/\mu]\, d\xi_1 \quad (30a)$$

$$Y(\mu, \xi_L) = \exp[-\xi_L/\mu] + \int_0^{\xi_L} \Phi(\xi_L - \xi_1) \exp[-\xi_1/\mu]\, d\xi_1 \quad (30b)$$

The auxiliary functions $X(\mu, \xi_L)$ and $Y(\mu, \xi_L)$ have been calculated by Chandrasekhar and Elbert [11, 12] and subsequent solutions are related to them. (In the Russian literature they are denoted as $\varphi(\mu, \xi_L)$ and $\psi(\mu, \xi_L)$ and called Ambartsumian's functions.) If the nth moments of X and Y are defined as

$$\alpha_n(\xi_L) = \int_0^1 X(\mu, \xi_L)\, \mu^n\, d\mu \quad (31a)$$

$$\beta_n(\xi_L) = \int_0^1 Y(\mu, \xi_L)\, \mu^n\, d\mu \quad (31b)$$

equation (29) takes the form

$$\alpha_0 + \beta_0 = 2 \quad (32)$$

If equation (28) is integrated with respect to ξ between 0 and ξ_L, one has

$$\int_0^1 d\mu \int_0^{\xi_L} F(\xi,\mu)\, d\xi = \xi_L \tag{33}$$

Multiplying equation (21) by $\exp[-\xi_1/\mu]$ and integrating with respect to ξ_1 between 0 and ξ_L, we get

$$\frac{\partial F}{\partial \xi} = -\frac{1}{\mu} F(\xi,\mu) + X(\mu,\xi_L)\,\Phi(\xi) - Y(\mu,\xi_L)\,\Phi(\xi_L-\xi)$$
$$= -\frac{1}{\mu} F(\xi,\mu) + \tfrac{1}{2} X(\mu,\xi_L) \int_0^1 F(\xi,\mu)\frac{d\mu}{\mu}$$
$$- \tfrac{1}{2} Y(\mu,\xi_L) \int_0^1 F(\xi_L-\xi,\mu)\frac{d\mu}{\mu} \tag{34}$$

A further integration with respect to ξ then gives

$$\int_0^{\xi_L} F(\xi,\mu)\frac{d\xi}{\mu} = [X(\mu,\xi_L) - Y(\mu,\xi_L)]\left[1 + \tfrac{1}{2}\int_0^1 d\mu \int_0^{\xi_L} F(\xi,\mu)\frac{d\xi}{\mu}\right] \tag{35}$$

Since the double integral on the right can be reproduced by a μ-wise integration of the same equation, one has

$$\int_0^{\xi_L} F(\xi,\mu)\frac{d\xi}{\mu} = \frac{X(\mu,\xi_L) - Y(\mu,\xi_L)}{1 - \frac{1}{2}(\alpha_0 - \beta_0)} \tag{36}$$

Equations (36) and (33) give, finally,

$$\alpha_1 - \beta_1 = \xi_L[1 - \tfrac{1}{2}(\alpha_0 - \beta_0)] = \beta_0 \xi_L \tag{37}$$

The functions $X(\mu, \xi_L)$ and $Y(\mu, \xi_L)$ are actually calculated as solutions of coupled, nonlinear integral equations [see 2 or 4]. A nonuniqueness actually arises in the calculations and is associated apparently with the conservative nature of the radiation field. Equation (37) provides a normalizing relation that is used to fix the proper physical solutions. It should be remarked that in [12] an asterisk is used to distinguish the proper functions from a second tabulated pair with an arbitrarily imposed normalization.

Equation (23) relates $L(\xi_1, \xi; \xi_L)$ to $\Phi(\xi)$ and it remains to express the latter in terms of the X, Y functions. This follows after a double integration of equation (34). The final relations are

$$\Phi(\xi) = M(\xi,\xi_L) + \int_0^{\xi} [\Phi(\xi_1)\, M(\xi-\xi_1,\xi_L) - \Phi(\xi_L-\xi_1)\, N(\xi-\xi_1,\xi_L)]\, d\xi_1 \tag{38a}$$

$$\Phi(\xi) = N(\xi_L-\xi,\xi_L) - \int_{\xi}^{\xi_L} [\Phi(\xi_1)\, M(\xi-\xi_1,\xi_L) - \Phi(\xi_L-\xi_1)\, N(\xi-\xi_1,\xi_L]\, d\xi_1 \tag{38b}$$

where

$$M(\xi,\xi_L) = \tfrac{1}{2}\int_0^1 X(\mu,\xi_L)\exp[-\xi/\mu]\frac{d\mu}{\mu}$$
$$N(\xi,\xi_L) = \tfrac{1}{2}\int_0^1 Y(\mu,\xi_L)\exp[-\xi/\mu]\frac{d\mu}{\mu} \tag{39}$$

Results directly applicable to the functions $\Theta(\xi)$ and $\Theta_s(\xi)$ now follow. In particular,

$$\Theta(\xi) = \tfrac{1}{2} + \frac{1}{2\Psi(\xi_L,\xi_L)}[\Psi(\xi_L-\xi,\xi_L) - \Psi(\xi,\xi_L)] \tag{40}$$

and

$$\Theta_s(\xi) = \frac{\Psi(\xi_L,\xi_L)}{4}[\Psi(\xi_L-\xi,\xi_L) + \Psi(\xi,\xi_L) - \Psi(\xi_L,\xi_L)] \tag{41}$$

where

$$\Psi(\xi, \xi_L) = 1 + \int_0^{\xi} \Phi(\xi_1)\, d\xi_1 \qquad (42)$$

and

$$\Psi(\xi_L, \xi_L) = 1/\beta_0 \qquad (43)$$

The wall values are

$$\Theta(0) = \alpha_0(\xi_L)/2, \quad \Theta(\xi_L) = \beta_0(\xi_L)/2 \qquad (44a)$$

$$\Theta_s(0) = \Theta_s(\xi_L) = 1/(4\beta_0) = (1/8)/[1 - \Theta(0)] \qquad (44b)$$

and the flux integrals yield

$$Q = 1 - 2\int_0^{\xi_L} \Theta(\xi_1)\, E_2(\xi_1)\, d\xi_1 = \beta_0(\alpha_1 + \beta_1) \qquad (45a)$$

$$Q_s = 2\int_0^{\xi_L} \Theta_s(\xi_1)\, E_2(\xi_1)\, d\xi_1 = \xi_L/2 \qquad (45b)$$

It follows from equations (44a) that the average of the two wall values is 1/2, since $\alpha_0 + \beta_0 = 2$. This conclusion follows independently from the more general result that $\Theta(\xi)$ is antisymmetric about the point $\xi/\xi_L = 1/2$, $\Theta(\xi_L/2) = 1/2$. The function $\Theta_s(\xi)$, on the other hand, is symmetric about $\xi/\xi_L = 1/2$. Table 1 gives values of α_0, $1/(4\beta_0)$, $\alpha_1 + \beta_1$, and $\beta_0(\alpha_1 + \beta_1)$ through the range $0 \leqslant \xi_L \leqslant 3$. These numerical results were derived from the work of Sobouti [13]. Numerical values for $\xi_L > 3$ are not included in the Table since the asymptotic expressions appearing at the bottom of the table are of comparable accuracy. These asymptotic forms will be discussed in the next section.

Equations (17) and (18) together with equations (45a) and (45b) permit ready calculation of temperature distributions and fluxes for arbitrary conditions. The functions $\Theta(\xi)$ and $\Theta_s(\xi)$ together with the appropriate values of $\alpha_n(\xi_L)$ and $\beta_n(\xi_L)$ are the requisite building blocks. An additional advantage arises from the fact that for $\xi_L \ll 1$ and $\xi_L \gg 1$ analytic expressions are available for the X and Y functions and therefore for their moments. These results can be used to calculate limiting forms of, say, flux and temperature slip. Before proceeding to these expressions it is convenient to list the equations of major interest, modified to conform with the final terminology:

Table 1

ξ_L	α_0	$(1/4\beta_0)$	$\alpha_1 + \beta_1$	$\beta_0(\alpha_1 + \beta_1)$
0·1	1·1419	0·2914	1·0672	0·9157
0·2	1·2228	0·3217	1·0926	0·8491
0·3	1·2838	0·3491	1·1080	0·7934
0·4	1·3331	0·3749	1·1185	0·7458
0·5	1·3746	0·3998	1·1259	0·7040
0·6	1·4103	0·4240	1·1316	0·6672
0·8	1·4692	0·4711	1·1392	0·6046
1·0	1·5163	0·5170	1·1440	0·5532
1·5	1·6024	0·6289	1·1501	0·4572
2·0	1·6615	0·7388	1·1525	0·3900
2·5	1·7051	0·8480	1·1538	0·3401
3·0	1·7386	0·9568	1·1542	0·3016
$\xi_L \gg 1$	$2 - \frac{2/(\sqrt{3})}{\gamma + \xi_L}$	$\frac{(\sqrt{3})}{8}(\gamma + \xi_L)$	$\frac{2}{\sqrt{3}}$	$\frac{4/3}{\gamma + \xi_L}$

$\gamma = 1{\cdot}42089$

$$\sigma T^4 - \sigma T_{w2}^4 = \frac{(\sigma T_{w1}^4 - \sigma T_{w2}^4)\left[\Theta(\xi) + \left(\frac{1}{\epsilon_{w2}} - 1\right)\beta_0(a_1+\beta_1)\right]}{1 + \left(\frac{1}{\epsilon_{w1}} + \frac{1}{\epsilon_{w2}} - 2\right)\beta_0(a_1+\beta_1)} + \frac{S}{k}\left\{\Theta_s(\xi) + \frac{\xi_L}{2}\frac{\left[\left(\frac{1}{\epsilon_{w1}} - 1\right) - \left(\frac{1}{\epsilon_{w2}} - 1\right)\right]\Theta(\xi) + \left(\frac{1}{\epsilon_{w2}} - 1\right) + 2\left(\frac{1}{\epsilon_{w1}} - 1\right)\left(\frac{1}{\epsilon_{w2}} - 1\right)\beta_0(a_1+\beta_1)}{1 + \left(\frac{1}{\epsilon_{w1}} + \frac{1}{\epsilon_{w2}} - 2\right)\beta_0(a_1+\beta_1)}\right\} \tag{46}$$

$$\left[\frac{\sigma T^4(\xi_L) - \sigma T_{w2}^4}{\sigma T_{w1}^4 - \sigma T_{w2}^4}\right]_{S=0} = \beta_0 \frac{\frac{1}{2} + \left(\frac{1}{\epsilon_{w2}} - 1\right)(a_1+\beta_1)}{1 + \left(\frac{1}{\epsilon_{w1}} + \frac{1}{\epsilon_{w2}} - 2\right)\beta_0(a_1+\beta_1)} \tag{47a}$$

$$\left[\frac{\sigma T_{w1}^4 - \sigma T^4(0)}{\sigma T_{w1}^4 - \sigma T_{w2}^4}\right]_{S=0} = \beta_0 \frac{\frac{1}{2} + \left(\frac{1}{\epsilon_{w1}} - 1\right)(a_1+\beta_1)}{1 + \left(\frac{1}{\epsilon_{w1}} + \frac{1}{\epsilon_{w2}} - 2\right)\beta_0(a_1+\beta_1)} \tag{47b}$$

$$\left[\frac{\sigma T^4(\xi_L) - \sigma T_{w2}^4}{S/k}\right]_{T_{w1}=T_{w2}} = \frac{1}{4\beta_0} + \frac{\xi_L}{2}\frac{\left(\frac{\beta_0}{2}\right)\left[\left(\frac{1}{\epsilon_{w1}} - 1\right) - \left(\frac{1}{\epsilon_{w2}} - 1\right)\right] + \left(\frac{1}{\epsilon_{w2}} - 1\right) + 2\left(\frac{1}{\epsilon_{w1}} - 1\right)\left(\frac{1}{\epsilon_{w2}} - 1\right)\beta_0(a_1+\beta_1)}{1 + \left(\frac{1}{\epsilon_{w1}} + \frac{1}{\epsilon_{w2}} - 2\right)\beta_0(a_1+\beta_1)} \tag{47c}$$

$$q_{w1} = q_{w2} - \frac{S}{k}\xi_L = \frac{(\sigma T_{w1}^4 - \sigma T_{w2}^4)\beta_0(a_1+\beta_1) - \frac{S}{k}\xi_L\left[\frac{1}{2} + \left(\frac{1}{\epsilon_{w2}} - 1\right)\beta_0(a_1+\beta_1)\right]}{1 + \left(\frac{1}{\epsilon_{w1}} + \frac{1}{\epsilon_{w2}} - 2\right)\beta_0(a_1+\beta_1)} \tag{48}$$

$$\left[\frac{q_{w1}}{\sigma T_{w1}^4 - \sigma T_{w2}^4}\right]_{S=0} = \frac{\beta_0(a_1+\beta_1)}{1 + \left(\frac{1}{\epsilon_{w1}} + \frac{1}{\epsilon_{w2}} - 2\right)\beta_0(a_1+\beta_1)} \tag{49a}$$

$$\left[\frac{q_{w1}}{S/k}\right]_{T_{w1}=T_{w2}} = \frac{-\xi_L\left[\frac{1}{2} + \left(\frac{1}{\epsilon_{w2}} - 1\right)\beta_0(a_1+\beta_1)\right]}{1 + \left(\frac{1}{\epsilon_{w1}} + \frac{1}{\epsilon_{w2}} - 2\right)\beta_0(a_1+\beta_1)} \tag{49b}$$

Equation (46) provides the additive components of $\sigma T^4 - \sigma T_{w2}^4$ that are attributable to differences in wall temperatures and source strength. The latter contribution is a coupling of the functions $\Theta(\xi)$ and $\Theta_s(\xi)$ except when the walls have equal emissivities. Equations (47) are exact forms of temperature slip for zero source strength and for equal wall temperatures, respectively. Equation (48) gives the complete expression for wall fluxes and equations (49) are the separate components from which total flux can be calculated.

Figures 4, 5, 6, and 7 were calculated from equations (49a), (49b), (47a), and (47c), respectively, for $\epsilon_{w1} = 0{\cdot}8$ and various values of ϵ_{w2}. From Sobouti's [13] tables of moments an accuracy to four decimal places is immediately possible.

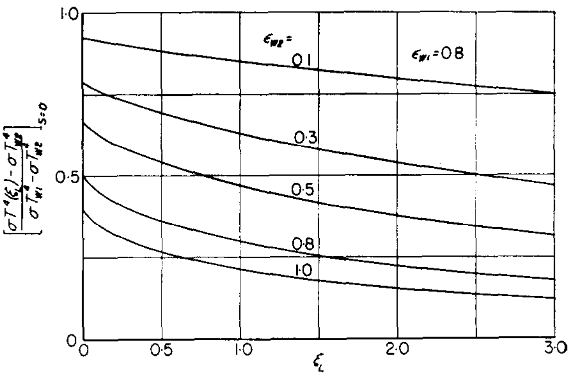

FIG. 6. Dimensionless temperature slip for zero source strength showing dependence on optical thickness and wall emissivities.

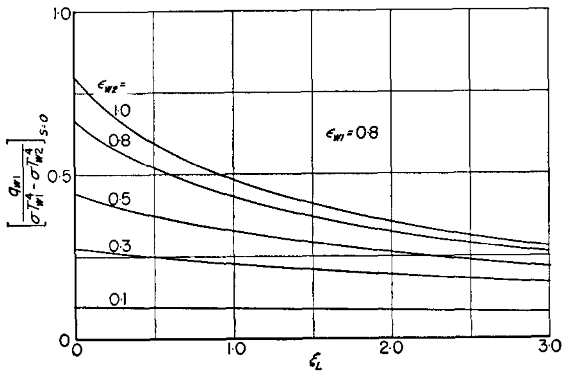

FIG. 4. Dimensionless flux for zero source strength showing dependence on optical thickness and wall emissivities.

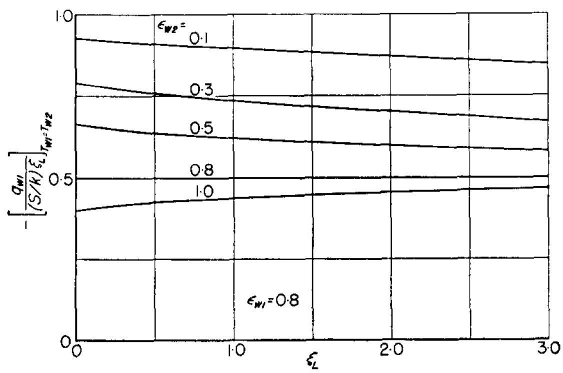

FIG. 5. Dimensionless flux for equal wall temperatures showing dependence on optical thickness and wall emissivities.

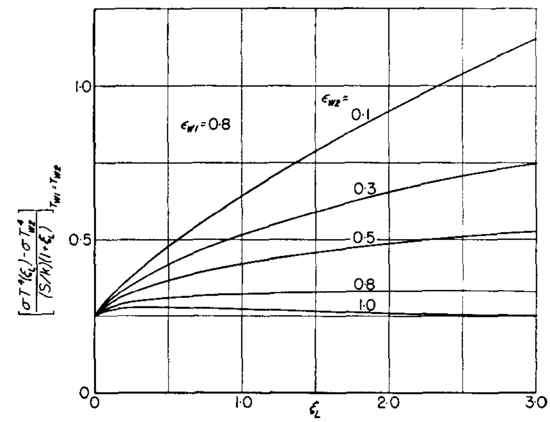

FIG. 7. Dimensionless temperature slip for equal wall temperature showing dependence on optical thickness and wall emissivities.

Limiting solutions

When optical thickness is infinitely large one has

$$X(\mu, \xi_L) \to H(\mu), \quad Y(\mu, \xi_L) \to 0, \quad \xi_L \to \infty \quad (50)$$

where $H(\mu)$ is the function introduced by Chandrasekhar [2] to study radiation transfer in a semi-infinite medium. At the other extreme of thickness a first approximation yields

$$X(\mu, \xi_L) \to 1, \quad Y(\mu, \xi_L) \to \exp[-\xi_L/\mu], \quad \xi_L \to 0 \quad (51)$$

Sobolev [14] has extended equation (50) as follows:

$$X(\mu, \xi_L) \sim H(\mu) - \mu H(\mu)/(\xi_L + \gamma) \quad (52a)$$

$$Y(\mu, \xi_L) \sim \mu H(\mu)/(\xi_L + \gamma), \quad \xi_L \gg 1 \quad (52b)$$

where γ is a constant equal to twice the ratio of the second and first moments of $H(\mu)$, Analysis of the semi-infinite case [see, e.g. 3] gives the values of 2 and $2/\sqrt{3}$, respectively, for the zeroth and first moments of $H(\mu)$ and $\gamma = 1{\cdot}420892$. It follows, after integration of equations (52), that

$$2 - \alpha_0 = \beta_0 \sim (2/\sqrt{3})/(\gamma + \xi_L) \quad (53a)$$

$$\alpha_1 + \beta_1 \sim 2/\sqrt{3} \quad (53b)$$

From [2, p. 204], the corrected forms of equations (51) are

$$X(\mu, \xi_L) \approx 1 + \frac{1 - E_2(\xi_L)}{1 - 2E_3(\xi_L)}\mu(1 - \exp[-\xi_L/\mu]) \quad (54a)$$

$$Y(\mu, \xi_L) \approx \exp[-\xi_L/\mu] + \frac{1 - E_2(\xi_L)}{1 - 2E_3(\xi_L)}\mu(1 - \exp[-\xi_L/\mu]),\ \xi_L \ll 1 \quad (54b)$$

and, after integration,

$$2 - \alpha_0 = \beta_0 \approx [1 + E_2(\xi_L)]/2 \quad (55a)$$

$$\alpha_1 + \beta_1 \approx \tfrac{1}{2}[1 + 2E_3(\xi_L)] + \tfrac{2}{3}\frac{[1 - E_2(\xi_L)][1 - 3E_4(\xi_L)]}{[1 - 2E_3(\xi_L)]} \quad (55b)$$

Figures 8 and 9 have been drawn with an expanded scale to show how well the approximations of equations (53) and (55) predict the flux and temperature slip for black walls and $S = 0$. A striking feature is the accuracy with which the asymptotic expression for $\beta_0(\alpha_1 + \beta_1)$ predicts flux through the complete range $0 \leqslant \xi_L \leqslant \infty$. From equations (53) one has

$$\beta_0(\alpha_1 + \beta_1) \sim \frac{1}{1{\cdot}06567 + \frac{3}{4}\xi_L} \quad (56)$$

and the error incurred at $\xi_L = 0$ is only 6 per cent. Since from equations (55), $\beta_0(\alpha_1 + \beta_1) = 1$ at $\xi_L = 0$, one concludes that

$$\beta_0(\alpha_1 + \beta_1) \sim \frac{1}{1 + \frac{3}{4}\xi_L} \quad (57)$$

must provide a good interpolation formula. Corresponding to this approximation one has, in general,

$$\left[\frac{q_{w1}}{\sigma T_{w1}^4 - \sigma T_{w2}^4}\right]_{S=0} = \frac{1}{(1/\epsilon_{w1}) + (1/\epsilon_{w2}) - 1 + \frac{3}{4}\xi_L} \quad (58a)$$

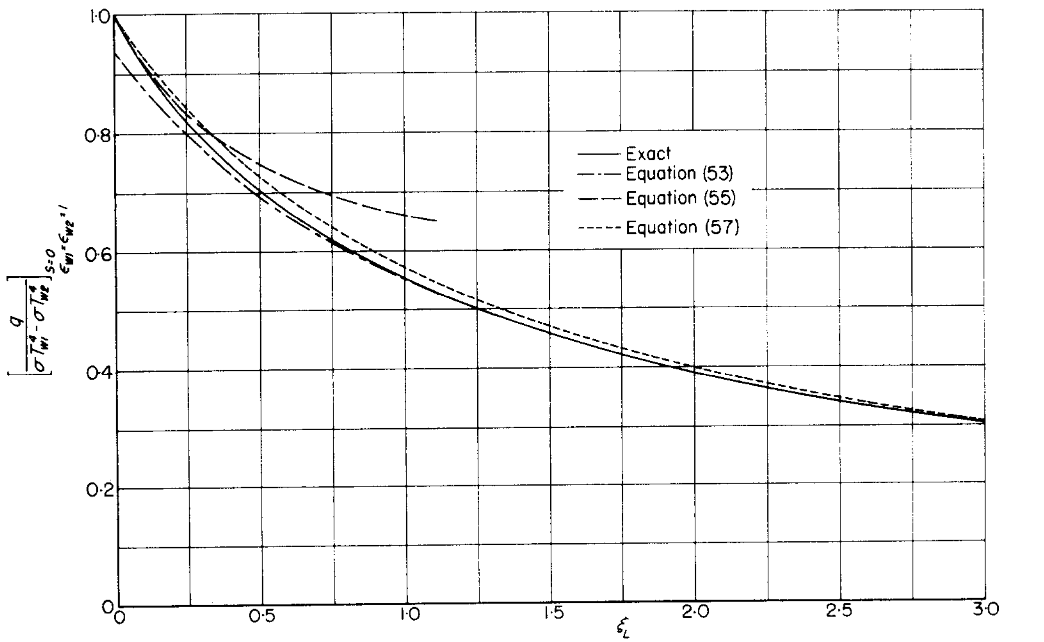

FIG. 8. Comparison between exact and approximate predictions of flux for black walls and zero source strength.

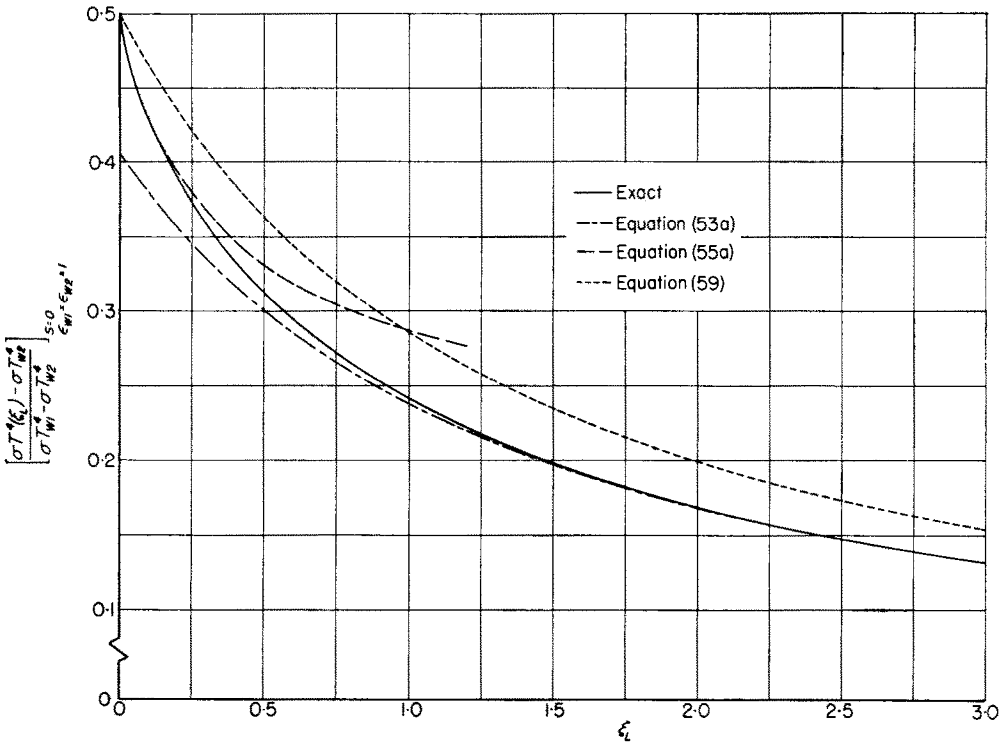

FIG. 9. Comparison between exact and approximate predictions of temperature slip for black walls and zero source strength.

$$\left[\frac{q_{w1}}{S/k}\right]_{T_{w1}=T_{w2}} = -\xi_L \frac{\dfrac{1}{\epsilon_{w2}} - \frac{1}{2} + \frac{3}{8}\xi_L}{\dfrac{1}{\epsilon_{w1}} + \dfrac{1}{\epsilon_{w2}} - 1 + \frac{3}{4}\xi_L} \tag{58b}$$

Equation (58a) has been proposed by many authors [see, e.g. 15, 8, 16] and its predictions for $\epsilon_{w1} = \epsilon_{w2} = 1$ are included in Fig. 8. A further discussion of the approximation is given in the next section. It suffices here to remark that the predictions may be acceptable for many purposes. The maximum error is about 3 per cent and occurs near $\xi_L = 0{\cdot}4$.

Figure 9 shows, as might be expected, that the asymptotic predictions of temperature slip deviate more drastically from the exact predictions for decreasing values of optical thickness. Equations (55) reveal that the slope of the curve becomes logarithmically infinite at $\xi_L = 0$. An algebraic interpolation formula is thus possible only in a very restricted sense. The figure shows a plot of the relation

$$\left[\frac{\sigma T^4(\xi_L) - \sigma T_{w2}^4}{\sigma T_{w1}^4 - \sigma T_{w2}^4}\right]_{\substack{S=0\\ \epsilon_{w1}=\epsilon_{w2}=1}} = \frac{\frac{1}{2}}{1 + \frac{3}{4}\xi_L} \tag{59}$$

which is a result of the special approximation discussed in the next section. The true value of temperature slip is achieved at $\xi_L = 0$ and for infinitely large values of ξ_L the function vanishes. The rates of change with ξ_L are, however, incorrect in both limiting cases and a serious loss of accuracy occurs in the transition region.

Approximations from interpolation formula

Equation (57) has been shown to lead to an unusually good interpolation formula for flux. Even though a comparable degree of excellence cannot be maintained in estimating temperature slip, it is possible to use the interpolation to derive temperature distributions which will maintain a fair accuracy except in the vicinity of the walls. Such an approach is considered here. The development is intimately related to the so-called diffusion approximation [see 17, 8, 9] which, in turn, utilizes the Eddington [18, p. 101] relation

$$-\frac{4}{3k}\frac{\mathrm{d}\sigma T^4}{\mathrm{d}x} = q \tag{60}$$

for flux. Equation (60) is an asymptotic relation, $\xi \gg 1$, and obviously suffers its greatest inaccuracy in the immediate vicinity of a boundary.

From equation (60) and equations (58) one has

$$\frac{\mathrm{d}\Theta}{\mathrm{d}\xi} = -\tfrac{3}{4}\frac{1}{1+\tfrac{3}{4}\xi_L} \tag{61a}$$

$$\frac{\mathrm{d}^2\Theta_s}{\mathrm{d}\xi^2} = \tfrac{3}{4} \tag{61b}$$

Equation (61a) yields

$$\Theta(\xi) = -\tfrac{3}{4}\frac{\xi}{1+\tfrac{3}{4}\xi_L} + A \tag{62a}$$

and equation (61b) gives

$$\Theta_s(\xi) = -\tfrac{3}{8}\left(\xi - \frac{\xi_L}{2}\right)^2 + B \tag{62b}$$

where A and B are constants of integration and in the latter equation use is made of the known symmetry of $\Theta_s(\xi)$ about $\xi_L/2$.

The single constant appearing in each of equations (62) limits the flexibility of the results. Thus, we are not able to use all the available information. Here, the flux integrals in equations (45) will be used to find A and B and comparisons will then be made with the true wall values given by equations (44).

If equation (62a) is substituted in the integrand of equation (45a) and the approximation of equation (57) is used, one has

$$1 + \frac{(3/2)}{1+\tfrac{3}{4}\xi_L}\left[\tfrac{1}{3} - \xi_L E_3(\xi_L) - E_4(\xi_L)\right] - 2A\left[\tfrac{1}{2} - E_3(\xi_L)\right] = \frac{1}{1+\tfrac{3}{4}\xi_L} \tag{63}$$

At $\xi_L = 0$, the equality holds, independent of A, and for $\xi_L \gg 1$

$$1 + \frac{\tfrac{1}{2}}{1+\tfrac{3}{4}\xi_L} - A = \frac{1}{1+\tfrac{3}{4}\xi_L}$$

from which A follows as a function of ξ_L. One thus gets

$$\Theta(\xi) = \frac{-\tfrac{3}{4}(\xi - \xi_L) + \tfrac{1}{2}}{1+\tfrac{3}{4}\xi_L} \tag{64}$$

We observe that $\Theta(\xi_L/2) = \tfrac{1}{2}$ which is exact; also

$$\Theta(0) = 1 - \frac{\tfrac{1}{2}}{1+\tfrac{3}{4}\xi_L} = 1 - \Theta(\xi_L) \tag{65}$$

Since $\Theta(0) = 1 - \tfrac{1}{2}\beta_0(\xi_L)$, the approximation imposes the condition

$$\beta_0(\xi_L) \doteq \frac{1}{1+\tfrac{3}{4}\xi_L}$$

and this must be compared with the exact relations

$$\beta_0(0) = 1, \quad \beta_0(\xi_L) \sim \frac{1{\cdot}155}{1{\cdot}421 + \xi_L} \tag{66}$$

which follow from equations (55a) and (53a). A considerable loss in accuracy is obviously incurred if equation (64) is used to determine $\Theta(0)$.

Equation (64) predicts the temperature slip given in equation (59) and plotted in Fig. 9. Together with equation (46) it leads to

$$\left[\frac{\sigma T^4 - \sigma T_{w2}^4}{\sigma T_{w1}^4 - \sigma T_{w2}^4}\right]_{S=0} = \frac{-\tfrac{3}{4}(\xi - \xi_L) + ([1/\epsilon_{w2}) - \tfrac{1}{2}]}{(1/\epsilon_{w1}) + (1/\epsilon_{w2}) - 1 + \tfrac{3}{4}\xi_L} \tag{67}$$

which is equivalent to the diffusion approximation of Howell and Perlmutter [8, equation (58)].

If equation (62b) is used together with equation (45b), one has

$$-\tfrac{3}{4}\left[\tfrac{1}{2} - \tfrac{2}{3}\left(\frac{\xi_L}{2}\right) + \tfrac{1}{2}\left(\frac{\xi_L}{2}\right)^2 - \left(\frac{\xi_L}{2}\right)^2 E_3(\xi_L) - 2\left(\frac{\xi_L}{2}\right)E_4(\xi_L) - 2E_5(\xi_L)\right] + 2B\left[\tfrac{1}{2} - E_3(\xi_L)\right] = \frac{\xi_L}{2} \tag{68}$$

This equality holds, independent of B, when $\xi_L = 0$, and for $\xi_L \gg 1$ becomes

$$-\tfrac{3}{4}\left[\tfrac{1}{2} - \tfrac{2}{3}\left(\frac{\xi_L}{2}\right) + \tfrac{1}{2}\left(\frac{\xi_L}{2}\right)^2\right] + B = \frac{\xi_L}{2}$$

The resulting expression for B gives

$$\Theta_s(\xi) = -\tfrac{3}{8}(\xi^2 - \xi\xi_L) + \frac{\xi_L}{4} + \tfrac{3}{8} \tag{69}$$

which agrees with Howell and Perlmutter's diffusion approximation [8, equation (59), $\epsilon_{w2} = 1$].

It remains to see how well equation (69) estimates $\Theta_s(0)$ or $\Theta_s(\xi_L)$. Since from equation (44b) $\Theta_s(0) = 1/[4\beta_0(\xi_L)]$, we have the exact conditions

$$\frac{1}{4\beta_0(0)} = \tfrac{1}{4}, \quad \frac{1}{4\beta_0(\xi_L)} \sim \frac{1{\cdot}421 + \xi_L}{4{\cdot}6} \sim \frac{\xi_L}{4{\cdot}6}$$

On the other hand, equation (69) is consistent with

$$\frac{1}{4\beta_0(\xi_L)} \doteq \tfrac{3}{8} + \frac{\xi_L}{4} \tag{70}$$

The error in the approximate wall values is more serious here than in the previous case since the magnitude of $\Theta_s(0)$ is incorrect at $\xi_L = 0$ and the linear increase with ξ_L for large optical thickness is incorrectly estimated. This discrepancy in rate cannot be modified in the diffusion approximation if proper flux is to result.

The previous development provides some insight into the usefulness and validity of the diffusion approximations. It is adapted to yield a good approximation of flux throughout the full range of optical thickness and gives reasonable temperature estimates in the portion of the medium removed from the walls. Temperature discontinuity at the wall is given exactly by the zeroth moments of Chandrasekhar's functions and serious discrepancies exist between the exact results and the predictions of diffusion theory. It has been noted that the flux relation of diffusion theory is a simple algebraic interpolation formula that is fitted to extreme values of ξ_L. It is possible to give similar results for temperature slip; namely,

$$\left[\frac{\sigma T^4(\xi_L) - \sigma T_{w2}^4}{\sigma T_{w1}^4 - \sigma T_{w2}^4}\right]_{\substack{S=0\\ \epsilon_{w1}=\epsilon_{w2}=1}} = \frac{1}{2 + (\sqrt{3})\,\xi_L}$$

$$\left[\frac{\sigma T^4(\xi_L) - \sigma T_{w2}^4}{S/k}\right]_{\substack{T_{w1}=T_{w2}\\ \epsilon_{w1}=\epsilon_{w2}=1}} \doteq \frac{(\sqrt{3})}{8}\,\xi_L + \tfrac{1}{4}$$

The desirability of using these simplified expressions must be based on the objectives at hand and the merits of having algebraic relations for ready calculation. For more exact predictions, the tabulated and graphical values of $\alpha_0(\xi_L)$ and $\beta_0(\xi_L)$ are available.

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Résumé—On a étudié le transport de chaleur par rayonnement à travers un gaz gris non isotherme absorbant et émetteur entre des parois chauffées. On a spécialement fait attention à l'évaluation de la température près des parois et du flux d'énergie par les méthodes et les fonctions tabulées étudiées par Chandrasekhar et Ambartsumian. La précision obtenue permet de se rendre compte de la précision des méthodes approchées existantes et des erreurs qui se produisent dans les solutions numériques des équations fondamentales.

Zusammenfassung—Zwischen beheizten Wänden wird der Wärmeübergang durch Strahlung durch ein nichtisothermes, absorbierendes und emittierendes graues Gas untersucht. Besondere Beachtung dabei findet die Berechnung der wandnahen Temperatur und die genaue Bestimmung der Energiestromdichte nach von Chandrasekhar und Ambartsumian untersuchten Methoden und tabellierten Funktionen. Die so erreichte Genauigkeit erlaubt eine Beurteilung der Genauigkeit von bestehenden Näherungsverfahren und der Fehler, die in numerische Lösungen der massgebenden Gleichungen eingehen.

Аннотация—Изучается радиационный теплоперенос через неизотермический поглощающий и излучающий серый газ между нагретыми стенками. Особое внимание направлено на определение температуры у стенок и на точное определение потока энергии по методу и через табулированные функции, изученные Чандрасекхаром и Абрамцумяном. Достигнутая точность позволяет оценить точность существующих приближенных методов, а также погрешности, встречающиеся в численных решениях основны хуравнений.